JOURNAL OF FUNCTIONAL ANALYSIS 57, 21–30 (1984)

Models for n-Tuples of Noncommuting Operators*

JOHN W. BUNCE

University of Kansas, Lawrence, Kansas 66045

Communicated by the Editors

Received September 7, 1983

This paper proves versions of the Rota model theorem, the de Branges–Rovnyak model theorem, and the coisometric extension theorem for n-tuples of not necessarily commuting operators. This generalizes the work of A. E. Frazho (*J. Funct. Anal.* **48** (1982), 1–11) for pairs of operators. The methods involve applying the single operator results to matrices of operators.

In [6], Frazho developed a model theory for certain pairs of noncommuting contraction operators on Hilbert space. He proved, for pairs of contractions, versions of the Rota model theorem, the de Branges–Rovnyak model theorem, and the coisometric extension theorem. Frazho proved his results by constructing pairs of shift operators on a Fock space. In this note we apply the known results for single operators to an operator-valued matrix and obtain most of his results for n-tuples of noncommuting operators. However, our models are not exhibited as concretely as his.

We begin with a coisometric extension theorem which generalizes Proposition 4 of [6]. Let Λ denote either the set of all natural numbers or the set $\{1, 2, 3,..., n\}$ for some natural number n.

PROPOSITION 1. *Let $\{A_i : i \in \Delta\}$ be a family of bounded linear operators on a Hilbert space H. Then the following two conditions are equivalent.*

(1) $\sum_{i\in\Lambda} A_i^* A_i \leqslant I$.

(2) *There exists a Hilbert space K containing H and coisometries $\{S_i : i \in \Lambda\}$ acting on K such that $S_i S_j^* = 0$ for $i \neq j$, and $S_i(H) \subseteq H$, $S_i|_H = A_i$ for each i.*

Proof. First, assume that the coisometries $\{S_i\}$ exist as in (2). The family $\{S_i^* S_i : i \in \Lambda\}$ is an orthogonal family of projections on K, so

$$Q = \sum_{i\in\Lambda} S_i^* S_i \leqslant I.$$

* This research was partially supported by NSF Grant MCS-8002138.

0022-1236/84 $3.00

Let P be the projection of K onto H. Then

$$\begin{aligned}\sum_{i\in\Lambda} A_i^*A_i &= \sum PS_i^*S_i|_H\\ &= PQP|_H \leqslant I.\end{aligned}$$

Conversely, assume that

$$\sum_{i\in\Lambda} A_i^*A_i \leqslant I.$$

Let $H_\Lambda = \sum_{i\in\Lambda} \oplus H$ be the Hilbert space direct sum of cardinality (Λ) many copies of H. Let $T\in B(H_\Lambda)$ be defined by

$$T(x_1, x_2, x_3,...) = (A_1x_1, A_2x_1, A_3x_1,...);$$

that is, T is an operator-valued matrix with the A_i down the first column and zeros elsewhere. Then T^*T is the operator matrix with $\sum_{i\in\Lambda} A_i^*A_i$ in the upper left-hand corner and zeroes elsewhere, so $T^*T\leqslant I$, and T is a contraction. Then by the coisometric extension theorem (see [5, p. 49]) there is a Hilbert space K_0 containing H_Λ, and a coisometry $S\in B(K_0)$ with $S(H_\Lambda)\subseteq H_\Lambda$ and $S|_{H_\Lambda} = T$. Let L be the orthogonal complement of H_Λ in K_0, $K_0 = H_\Lambda \oplus L$. With respect to this decomposition of K_0, the matrix of S is

$$S = \begin{pmatrix} A_1 & 0 & 0 & \cdots & X_1 \\ A_2 & 0 & 0 & \cdots & X_2 \\ \vdots & \ddots & & & \vdots \\ 0 & 0 & 0 & \cdots & Y \end{pmatrix},$$

where, for $i\in\Lambda$, $X_i : L\to H$, and $Y: L\to L$. Since $SS^* = I$ it is easily seen that $A_iA_i^* + X_iX_i^* = I$ for $i\in\Lambda$, $A_iA_j^* + X_iX_j^* = 0$ for $i\neq j$, $X_iY^* = 0$ for $i\in\Lambda$, and $YY^* = I$.

If $Y^*Y = I$, then all $X_i = 0$ and the A_i are coisometries with orthogonal initial spaces and the conclusion is trivial. Otherwise, $Y^*Y\neq I$ and L must be infinite dimensional. There then exists a family $\{Z_i : i\in\Lambda\}$ of coisometries acting on L such that the Z_i have orthogonal initial spaces; $Z_iZ_i^* = I$ for $i\in\Lambda$ and $Z_iZ_j^* = 0$ for $i\neq j$. Let $K = H\oplus L$ and define $S_i\in B(K)$ by

$$S_i = \begin{pmatrix} A_i & X_i \\ 0 & Z_iY \end{pmatrix}.$$

An easy computation shows that the S_i are coisometries with orthogonal initial spaces and completes the proof.

We remark that the pair of coisometries constructed by Frazho in [6, Proposition 4] have orthogonal initial spaces, although Frazho did not state it. In [4], Durszt and Sz.-Nagy proved that if $\{A_\alpha\}$ is an arbitrary family of contractions acting on H, then there is a Hilbert space K containing H and a family $\{S_\alpha\}$ of coisometries on K such that each S_α extends A_α.

Let $r(T)$ denote the spectral radius of an operator T. The following theorem is a version of the de Branges–Rovnyak model theorem, see [5, p. 23]. One would like to replace the condition that each A_i have spectral radius less than one (i.e., A_i^n converges to zero in norm for each i) by the condition that A_i^n converges to zero strongly for each i; however, we have not been able to prove the theorem in that case. Recall that a coisometry S is called pure if S^n converges to zero strongly (see [5, Sect. 1]).

PROPOSITION 2. *For n a natural number let $\{A_i : 1 \leqslant i \leqslant n\} \subseteq B(H)$ be such that $r(A_i) < 1$ for each i and $\sum_{i=1}^n A_i^* A_i \leqslant I$. Then there is a Hilbert space K containing H and pure coisometries $\{S_i : 1 \leqslant i \leqslant n\}$ acting on K such that $S_i(H) \subseteq H$, $S_i|_H = A_i$ for each i, and $S_i S_j^* = 0$ for $i \neq j$.*

Proof. Let $H^{(n)}$ be the direct sum of n copies of H. As in the proof of Proposition 1, define $T \in B(H^{(n)})$ by $T(x_1, x_2, ..., x_n) = (A_1 x_1, A_2 x_1, ..., A_n x_1)$. Then T is a contraction, and T^k converges to zero strongly since A_1^k converges to zero strongly. The de Branges–Rovnyak theorem then implies that there is a Hilbert space K_0 containing $H^{(n)}$ and a pure coisometry S on K_0 such that $S(H^{(n)}) \subseteq H^{(n)}$ and $S|_H(n) = T$. Let L be the orthogonal complement of $H^{(n)}$ in K_0. Then, as in the proof of Proposition 1, the matrix of S with respect to the decomposition of K_0 as $H^{(n)} \oplus L$ is

$$S = \begin{pmatrix} A_1 & 0 & 0 & \cdots & X_1 \\ A_2 & 0 & 0 & \cdots & X_2 \\ \vdots & & & & \vdots \\ A_n & 0 & 0 & \cdots & X_n \\ 0 & 0 & 0 & \cdots & Y \end{pmatrix},$$

and again $A_i A_i^* + X_i X_i^* = I$, $A_i A_j^* + X_i X_j^* = 0$ for $i \neq j$, $X_i Y^* = 0$, and $YY^* = I$. Since S is pure, Y^k converges to zero strongly. If $L = \{0\}$, then A_i would be a coisometry of spectral radius less than one, so $L \neq \{0\}$. So Y is a pure coisometry and there exists a Hilbert space M such that

$$L = l^2(M) = \left\{ (x_1, x_2, ...): x_i \in M, \sum \|x_i\|^2 < \infty \right\}$$

and $Y(x_1, x_2, ...) = (x_2, x_3, ...)$. Define $Z_i \in B(L)$ for $1 \leqslant i \leqslant n$ by

$$Z_i(x_1, x_2, x_3, ...) = (x_i, x_{n+i}, x_{2n+i}, ...).$$

Then each Z_i is a coisometry, and the Z_i have orthogonal initial spaces. Since

$$(Z_i Y)(x_1, x_2, \ldots) = (x_{i+1}, x_{i+1+n}, x_{i+1+2n}, \ldots),$$

it is easy to see that $(Z_i Y)^k$ converges strongly to zero for each i. Let $K = H \oplus L$ and define $S_i \in B(K)$ by

$$S_i = \begin{pmatrix} A_i & X_i \\ 0 & Z_i Y \end{pmatrix}.$$

Then the S_i are coisometries with orthogonal initial spaces. The proof will be complete after we show that S_i^k converges strongly to zero for each i. A computation shows that

$$S_i^k = \begin{pmatrix} A_i^k & \sum_{j=0}^{k} A_i^j X_i (Z_i Y)^{k-j} \\ 0 & (Z_i Y)^k \end{pmatrix}.$$

Since A_i^k converges to zero in norm and $(Z_i Y)^k$ converges to zero strongly, it suffices to show that

$$\left\| \sum_{j=0}^{k} A_i^j X_i (Z_i Y)^{k-j} y \right\| \to 0$$

for each $y \in L$. To this end, fix y and let $\varepsilon > 0$ be given. There exists an N such that $\|(Z_i Y)^j y\| < \varepsilon$ for all $j \geqslant N$. Then if $k \geqslant N$, $k = N + m$, $m \geqslant 0$, we have

$$\begin{aligned} &\left\| \sum_{j=0}^{k} A_i^j X_i (Z_i Y)^{k-j} y \right\| \\ &\quad \leqslant \left\| \sum_{j=0}^{m} A_i^j X_i (Z_i Y)^{N+(m-j)} y \right\| + \left\| \sum_{j=m+1}^{k} A_i^j X_i (Z_i Y)^{k-j} y \right\| \\ &\quad \leqslant \left(\sum_{j=0}^{m} \|A_i^j\| \right) \varepsilon + \left(\sum_{j=m+1}^{m+N} \|A_i^j\| \right) \|y\| \\ &\quad \leqslant \left(\sum_{j=0}^{\infty} \|A_i^j\| \right) \varepsilon + \left(\sum_{j=m+1}^{\infty} \|A_i^j\| \right) \|y\|. \end{aligned}$$

But $r(A_i) < 1$, so then by the spectral radius formula and the root test, the infinite series in the first summand converges while the second summand approaches zero. Hence, S_i^k converges to zero strongly and the proof is complete.

The following theorem is a generalization of Rota's model theorem (for single operators) and [6, Proposition 1 and Remark, p. 6] (for pairs of operators). After stating and proving the theorem, we discuss in what way the hypothesis of the theorem is analogous to requiring that "the spectral radius be less than one."

PROPOSITION 3. *If $A_1, A_2, \ldots, A_n$ are in $B(H)$ and there exists a positive operator P in $B(H)$ such that*

$$\left(\sum_{i=1}^{n} A_i^* P A_i\right) + I = P, \tag{$*$}$$

then there exists a Hilbert space K containing H, a family $S_1, S_2, \ldots, S_n$ of pure coisometries with orthogonal initial spaces acting on K with $S_i(H) \subseteq H$ for each i, and an invertible operator $R \in B(H)$ such that $A_i = R^{-1}(S_i|_H)R$ for each i.

Proof. Since $I \leqslant P$, P is invertible. Let R be the positive square root of P and let $B_i = RA_iR^{-1}$. Then

$$\begin{aligned}\sum_{i=1}^{n} B_i^* B_i &= \sum R^{-1} A_i^* R^2 A_i R^{-1} \\ &= R^{-1}\left(\sum A_i^* P A_i\right) R^{-1} \\ &= R^{-1}(P - I)R^{-1} \\ &= I - P^{-1} \leqslant I.\end{aligned}$$

Let λ be in the approximate point spectrum of a fixed A_i and let $\{x_k\}$ be a sequence of unit vectors in H such that $\|(A_i - \lambda)x_k\| \to 0$. Let Lim be any generalized limit on the space of bounded complex sequences (see, e.g., [8, p. 104]). Then

$$\operatorname{Lim}(Px_k, x_k) = 1 + \operatorname{Lim}(A_i^* P A_i x_k, x_k) + \operatorname{Lim}\left(\sum{}' A_j^* P A_j x_k, x_k\right),$$

where the prime on the sum indicates that the ith term is omitted. Since $\|A_i x_k - \lambda x_k\| \to 0$, this becomes

$$\operatorname{Lim}(Px_k, x_k) = 1 + |\lambda|^2 \operatorname{Lim}(Px_k, x_k) + \operatorname{Lim}\left(\sum{}' A_j^* P A_j x_k, x_k\right),$$

so

$$(1 - |\lambda|^2) \operatorname{Lim}(Px_k, x_k) \geqslant 1$$

and $|\lambda|^2 < 1$. So the approximate point spectrum of A_i is inside the open unit ball, so $r(A_i) = r(B_i) < 1$. Hence, the B_i satisfy the hypotheses of Proposition 2, so there exists a Hilbert space K containing H and pure coisometries with orthogonal initial spaces $S_1, S_2, \ldots, S_n$ acting on K with $S_i(H) \subseteq H$ and $S_i|_H = B_i = RA_iR^{-1}$ for each i. Then $R^{-1}(S_i | H)R = A_i$, so the proof is complete.

If $n = 1$ in Proposition 3, then it follows from the proof that $r(A_1) < 1$. Conversely, if $r(A_1) < 1$ let

$$P = \sum_{i=0}^{\infty} A_1^{i*} A_1^i,$$

where the infinite sum converges in norm by the root test and the spectral radius formula. Then clearly

$$A_1^* P A_1 + I = P.$$

In this case, the operator R is precisely the R in the proof of [7, Lemma 1]. The rest of this paper examines condition (*) in the case $n > 1$.

We first define some notation. Let $A_1, A_2, \ldots, A_n$ be in $B(H)$. Let $F(k, n)$ be the set of all functions from the set $\{1, 2, 3, \ldots, k\}$ to the set $\{1, 2, 3, \ldots, n\}$. For f in $F(k. n)$, let

$$A_f = A_{f(1)} A_{f(2)} \cdots A_{f(k)}.$$

Our first lemma is merely bookkeeping.

LEMMA 4. (a) *If* $1 \leqslant m < k$,

$$\sum_{f \in F(k,n)} A_f^* A_f = \sum_{h \in F(k-m,n)} A_h^* \left(\sum_{g \in F(m,n)} A_g^* A_g \right) A_h.$$

(b) *For any* $m \geqslant 1$, $k \geqslant 1$,

$$\left\| \sum_{f \in F(mk,n)} A_f^* A_f \right\| \leqslant \left\| \sum_{g \in F(k,n)} A_g^* A_g \right\|^m.$$

Proof. Part (a) is obvious. For part (b) we compute

$$\begin{aligned} \left\| \sum_{f \in F(mk,n)} A_f^* A_f \right\| &= \left\| \sum_{h \in F((m-1)k,n)} A_h^* \left(\sum_{g \in F(k,n)} A_g^* A_g \right) A_h \right\| \\ &\leqslant \left\| \sum A_h^* \left(\left\| \sum_{g \in F(k,n)} A_g^* A_g \right\| \right) A_h \right\| \\ &= \left\| \sum_{g \in F(k,n)} A_g^* A_g \right\| \left\| \sum_{h \in F((m-1)k,n)} A_h^* A_h \right\|. \end{aligned}$$

An induction argument then completes the proof.

The proof of the following lemma is now almost identical to the proof of [1, Proposition 8]. We omit the proof.

LEMMA 5.

$$\lim_{k\to\infty} \left\| \sum_{f\in F(k,n)} A_f^* A_f \right\|^{1/k} = \inf_k \left\{ \left\| \sum_{f\in F(k,n)} A_f^* A_f \right\|^{1/k} \right\}.$$

For $A = (A_1, A_2, \ldots, A_n)$ *an* n-tuple *in* $B(H)$, *define*

$$r(A) = \inf_k \left\{ \left\| \sum_{h\in F(k,n)} A_f^* A_f \right\|^{1/(2k)} \right\}.$$

For $n = 1$, $r(A)$ *is the usual spectral radius of* A.

We now give some conditions equivalent to condition (*) in Proposition 3.

PROPOSITION 6. *Let* $A = (A_1, A_2, \ldots, A_n)$ *be an* n-tuple *in* $B(H)$. *Then the following conditions are equivalent.*

(a) *There is a positive operator* P *such that*

$$\left(\sum_{i=1}^{n} A_i^* P A_i \right) + I = P.$$

(b) *The infinite series*

$$\sum_{k=1}^{\infty} \left(\sum_{f\in F(k,n)} A_f^* A_f \right)$$

is convergent in the strong operator topology.

(c) $r(A) < 1$.

Proof. Assume condition (a) is true. Then $I \leqslant P$, so $A_i^* A_i \leqslant A_i^* P A_i$ and

$$\sum_{i=1}^{n} A_i^* A_i \leqslant \sum_{i=1}^{n} A_i^* P A_i = P - I.$$

So $I + \sum_{i=1}^{n} A_i^* A_i \leqslant P$. Now assume, as inductive hypothesis, that

$$I + \sum_{k=1}^{m} \left(\sum_{f\in F(k,n)} A_f^* A_f \right) \leqslant P.$$

We have just proved the case $m = 1$. If the equation is true for m, then, by multiplying on the left by A_j^*, on the right by A_j, and summing, we have that

$$\sum_{j=1}^{n} A_j^* A_j + \sum_{j=1}^{n} \sum_{k=1}^{m} \left(\sum_{f\in F(k,n)} (A_f A_j)^* A_f A_j \right) \leqslant \sum_{j=1}^{n} A_j^* P A_j = P - I.$$

So

$$I + \sum_{k=1}^{m+1} \left(\sum_{g \in F(k,n)} A_g^* A_g \right) \leqslant P.$$

Hence the infinite series in (b) is convergent in the strong operator topology.

Now assume the infinite series in (b) is convergent, and let

$$P = I + \sum_{k=1}^{\infty} \left(\sum_{f \in F(k,n)} A_f^* A_f \right).$$

Then

$$\begin{aligned}
\sum_{j=1}^{n} A_j^* P A_j &= \sum_{j=1}^{n} A_j^* A_j + \sum_{k=1}^{\infty} \left(\sum_{j=1}^{n} \left(\sum_{f \in F(k,n)} (A_f A_j)^* A_f A_j \right) \right) \\
&= \sum_{j=1}^{n} A_j^* A_j + \sum_{k=2}^{\infty} \sum_{g \in F(k,n)} A_g^* A_g \\
&= P - I,
\end{aligned}$$

so $I + \sum_{j=1}^{n} A_j^* P A_j = P$ and (b) implies (a). If $x \in H$, then

$$\|x\|^2 + \sum_{k-1}^{\infty} \left\langle \sum_{f \in F(k,n)} A_f^* A_f x, x \right\rangle = \langle Px, x \rangle,$$

where the angle brackets denote inner product, so

$$\left\| \left(\sum_{f \in F(k,n)} A_f^* A_f \right)^{1/2} x \right\| \to 0 \qquad \text{as} \quad k \to \infty.$$

Hence, by the uniform boundedness principle, there is a number M such that $\|\sum_{f \in F(k,n)} A_f^* A_f\| \leqslant M$ for all k. We now proceed as in [9, Proof of Theorem 5.1]. For $k = 0$ we define $\sum_{f \in F(k,n)} A_f^* A_f$ to be I. For $0 < r < (1/M^{1/2})$ define $m: H \to \{0, 1, 2, \ldots\}$ by

$$m(x) = \inf \left\{ k \geqslant 0 : \left\| \left(\sum_{f \in F(k,n)} A_f^* A_f \right)^{1/2} x \right\| \leqslant r \|x\| \right\}.$$

If $m(x) \geqslant 1$, then

$$\begin{aligned}
m(x) r^2 \|x\|^2 &\leqslant \sum_{k=0}^{m(x)-1} \left\| \left(\sum_{f \in F(k,n)} A_f^* A_f \right)^{1/2} x \right\|^2 \\
&\leqslant \sum_{k=0}^{\infty} \left\langle \sum_{f \in F(k,n)} A_f^* A_f x, x \right\rangle \\
&= (Px, x) \leqslant \|P\| \, \|x\|^2.
\end{aligned}$$

So $m(x) \leqslant \|P\|/r^2$. For $k > \|P\|/r^2$ we have, using Lemma 4,

$$\begin{aligned}
&\left\| \Big(\sum_{f \in F(k,n)} A_f^* A_f \Big)^{1/2} x \right\| \\
&\quad = \left\| \Big(\sum_{g \in F(m(x),n)} A_g^* \Big(\sum_{h \in F(k-m(x),n)} A_h^* A_h \Big) A_g \Big)^{1/2} x \right\| \\
&\quad \leqslant M^{1/2} \left\| \Big(\sum_{g \in F(m(x),n)} A_g^* A_g \Big)^{1/2} x \right\| \\
&\quad \leqslant M^{1/2} r \, \|x\|.
\end{aligned}$$

So $\|(\sum_{f \in F(k,n)} A_f^* A_f)^{1/2}\| \leqslant (M^{1/2} r) < 1$, and $\|\sum_{f \in F(k,n)} A_f^* A_f\| < 1$. It then follows from Lemma 5 that $r(A) < 1$. So (b) implies (c).

If $r(A) < 1$, then the series in (b) is even norm convergent by the root test. The proof of Proposition 6 is complete.

We remark that if $\sum_{j=1}^n A_j^* A_j \leqslant rI$, where $r < 1$, then the n-tuple $(A_1, A_2, ..., A_n)$ satisfies the hypothesis of Proposition 3 by Lemma 5 and Proposition 6. This is the content, in the case of $n = 2$, of [6, Corollary 1].

We recall (see [2, p. 500]) that if $A = (A_1, A_2, ..., A_n)$ is an n-tuple of operators acting on H, then the joint approximate point spectrum of A (also called the joint left spectrum of A), is the set of n-tuples of complex numbers $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$ such that the left ideal of $B(H)$ generated by the set $\{A_1 - \lambda_1 I, A_2 - \lambda_2 I, ..., A_n - \lambda_n I\}$ does not contain the identity operator. For λ an n-tuple of complex numbers let $\|\lambda\|$ be the Euclidean norm of λ.

PROPOSITION 7. *If $A = (A_1, A_2, ..., A_n)$ is an n-tuple of operators acting on H, then $\|\lambda\| \leqslant r(A)$ for any n-tuple λ in the joint approximate poin spectrum of A.*

Proof. If $\lambda = (\lambda_1, \lambda_2, ..., \lambda_n)$ is in the joint approximate point spectrum of A, then the left ideal generated by the set $\{A_i - \lambda_i I\}$ is proper so by [3, Theorem 2.9.5] there is a state ρ on the C^*-algebra $B(H)$ such that $\rho(XA_i) = \lambda_i \rho(X)$ for each X in $B(H)$ and each i. Then for any $k \geqslant 1$ $|\rho(\sum_{f \in F(k,n)} A_f^* A_f)| \leqslant \|\sum_{f \in F(k,n)} A_f^* A_f\|$, or

$$\sum_{f \in F(k,n)} \bar{\lambda}_f \lambda_f \leqslant \left\| \sum_{f \in F(k,n)} A_f^* A_f \right\|,$$

where $\lambda_f = \lambda_{f(1)} \lambda_{f(2)} \cdots \lambda_{f(n)}$. But

$$\begin{aligned}
\sum_{f \in F(k,n)} \bar{\lambda}_f \lambda_f &= \sum_{j=1}^n \bar{\lambda}_j \Big(\sum_{g \in F(k-1,n)} \bar{\lambda}_g \lambda_g \Big) \lambda_j \\
&= \Big(\sum_{j=1}^n |\lambda_j|^2 \Big) \Big(\sum_{g \in F(k-1,n)} \bar{\lambda}_g \lambda_g \Big),
\end{aligned}$$

so by induction

$$\sum_{f\in F(k,n)} \bar{\lambda}_f\lambda_f = \left(\sum_{j=1}^{n} |\lambda_j|^2\right)^k.$$

Hence

$$\left(\sum_{j=1}^{n} |\lambda_j|^2\right)^k \leqslant \left\| \sum_{f\in F(k,n)} A_f^* A_f \right\|$$

and $\|\lambda\| \leqslant r(A)$.

Although I have not been able to prove it, it seems likely that if $A = (A_1, A_2,..., A_n)$ is an n-tuple of commuting operators then

$$r(A) = \sup\{\|\lambda\| : \lambda \in a(A)\},$$

where $a(A)$ is the joint approximate point spectrum of the n-tuple A.

Note added in proof. Chandler Davis has informed me that Proposition 1 is a simple special case of the Naimark dilation theorem. A simple proof of Proposition 1 was given by Davis in [Some dilation and representation theorems, "Proceedings of the Second International Symposium in West Africa on Functional Analysis and Its Applications, Kumasi, 1979"].

References

1. F. F. Bonsall and J. Duncan, "Complete normed algebras," Springer-Verlag, Berlin, 1973.
2. John W. Bunce, The joint spectrum of commuting nonnormal operators, *Proc. Amer. Math. Soc.* **29** (1971), 499–505.
3. Jacques Dixmier, "C^*-algebras," North-Holland, Amsterdam, 1977.
4. E. Durszt and B. Sz.-Nagy, Remark to a paper of A. E. Frazho: "Models for noncommuting operators," *J. Funct. Anal.* **52** (1983), 146–147.
5. Peter A. Fillmore, "Notes on Operator Theory," Van Nostrand–Reinholds, New York, 1970.
6. Arthur E. Frazho, Models for noncommuting operators, *J. Funct. Anal.* **48** (1982), 1–11.
7. Frank Gilfeather, Norm conditions on resolvents of similarities of Hilbert space operators and applications to direct sums and integrals of operators, *Proc. Amer. Math. Soc.* **68** (1978), 44–48.
8. K. Yosida, "Functional Analysis," Springer-Verlag, New York, 1968.
9. J. Zabczyk, Remarks on the control of discrete-time distributed parameter systems *SIAM J. Control* **12** (1974), 721–735.